One important property related to spectrum of irrational number bigger 1.
https://www.linkedin.com/feed/update/urn:li:activity:7090662660945403904?utm_source=share&utm_medium=member_desktop
We define the spectrum of number $x \in \mathbb{R}$ as the following multiset
$Spec(x) = \{\lfloor x \rfloor, \lfloor 2x \rfloor, \lfloor 3x \rfloor, \lfloor 4x \rfloor, \ldots\}$.
Prove that $Spec\left(\sqrt{2}\right)$ and $Spec\left(2+\sqrt{2}\right)$ partition $\mathbb{N}$ i.e. that each natural number is an element of exactly one of these sets.
Solution by Arkady Alt, **San Jose**, **California**, **USA**.
Lemma.
If x irrational number and $x > 1$ then mapping $n \mapsto [nx] : \mathbb{N} \to \mathbb{N}$ is injection.
Proof.
Suppose that there are $n \neq m$ such that $\lfloor nx \rfloor = \lfloor mx \rfloor$ then $0 = \lfloor nx \rfloor - \lfloor mx \rfloor =$
$(n-m)x + \{mx\} - \{nx\} \iff (n-m)x = \{nx\} - \{mx\} \implies |n-m|x = |\{mx\} - \{nx\}|$.
Since $|\{mx\} - \{nx\}| < 1$ and $1 \leq |n-m|$ then $x \leq |n-m|x < 1$, i.e. contradiction.■
From the **Lemma** immediately follow that for any irrational $x > 1$ multiset $Spec(x)$ is a regular set and sequence $(\lfloor nx \rfloor)_{n\in\mathbb{N}}$ is strictly increasing.
Theorem.
If α and β are positive irrational numbers such that $\dfrac{1}{\alpha} + \dfrac{1}{\beta} = 1$ then

i. $Spec(\alpha) \cap Spec(\beta) = \varnothing$;
ii. $Spec(\alpha) \cup Spec(\beta) = \mathbb{N}$.

Proof.
First note that $\alpha, \beta > 0$ and $\dfrac{1}{\alpha} + \dfrac{1}{\beta} = 1$ implies $\alpha, \beta > 1$.
i. Suppose that there are two natural numbers n and m such that $[n\alpha] = [m\beta]$.
Let $p := [n\alpha] = [m\beta]$ then $\begin{cases} p < n\alpha < p+1 \\ p < m\beta < p+1 \end{cases} \iff \begin{cases} \dfrac{n}{p+1} < \dfrac{1}{\alpha} < \dfrac{n}{p} \\ \dfrac{m}{p+1} < \dfrac{1}{\beta} < \dfrac{m}{p} \end{cases} \implies$

$\dfrac{n}{p+1} + \dfrac{m}{p+1} < \dfrac{1}{\alpha} + \dfrac{1}{\beta} < \dfrac{n}{p} + \dfrac{m}{p} \iff \dfrac{n+m}{p+1} < 1 < \dfrac{n+m}{p} \implies p < n+m < p+1$.
But this is the contradiction, because interval $(p, p+1)$ not contains integer numbers.
ii. Assume that exist $p \in \mathbb{N} \setminus (Spec(\alpha) \cup Spec(\beta))$, i.e. $p \neq [n\alpha]$ and $p \neq [m\beta]$ for any $n, m \in \mathbb{N}$. Since sequences $(\lfloor n\alpha \rfloor)_{n\in\mathbb{N}}$ and $(\lfloor n\beta \rfloor)_{n\in\mathbb{N}}$ are strictly increasing then there are $n, m \in \mathbb{N}$ such that $\begin{cases} [n\alpha] < p < [(n+1)\alpha] \\ [m\beta] < p < [(m+1)\beta] \end{cases}$.
Since can't be $[n\alpha] < p < n\alpha$ because there is no integers between $[n\alpha]$ and $n\alpha$ we have $n\alpha < p < [(n+1)\alpha] \iff \begin{cases} n\alpha < p \\ p+1 \leq [(n+1)\alpha] \end{cases} \implies \begin{cases} n\alpha < p \\ p+1 < (n+1)\alpha \end{cases}$.
($p \neq [n\alpha], n \in \mathbb{N}$ by supposition and $p \neq n\alpha, n \in \mathbb{N}$ because α irrational).

Analogically, $[m\beta] < p < [(m+1)\beta] \Rightarrow \begin{cases} m\beta < p \\ p+1 < (m+1)\beta \end{cases}$.

Hence, $\frac{n}{p} < \frac{1}{\alpha} < \frac{n+1}{p+1}$ and $\frac{m}{p} < \frac{1}{\beta} < \frac{m+1}{p+1}$ and that implies

$\frac{n}{p} + \frac{m}{p} < \frac{1}{\alpha} + \frac{1}{\beta} < \frac{n+1}{p+1} + \frac{m+1}{p+1} \Leftrightarrow \frac{n+m}{p} < 1 < \frac{n+m+2}{p+1} \Leftrightarrow$

$\begin{cases} n+m < p \\ p+1 < n+m+2 \end{cases} \Leftrightarrow n+m < p < n+m+1$. That is the contradiction.■

In particular, since $\frac{1}{\sqrt{2}} + \frac{1}{2+\sqrt{2}} = 1$ then $Spec\left(\sqrt{2}\right) \cap Spec\left(2+\sqrt{2}\right) = \varnothing$ and $Spec\left(\sqrt{2}\right) \cup Spec\left(2+\sqrt{2}\right) = \mathbb{N}$.